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Nonlinear modal models for sonic fatigue response prediction: a comparison of methods

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Abstract

Accurate prediction of sonic fatigue response is important in designing aircraft structures for long life. Early prediction methods were based on single-mode, linear models which were not accurate for complex structures or large-amplitude response levels. Direct time integration of full, nonlinear, finite element models can provide accurate results, but at a prohibitive computational expense. Recent methods reduce the finite element model to a low-order system of nonlinear modal equations. The modal equations can then be integrated in the time domain. The computational burden is greatly reduced and an accurate response prediction can be accomplished. In this paper, several methods used to construct the nonlinear modal models are compared using a clamped–clamped beam as an example problem.

1. Introduction

Sonic fatigue is a maintenance problem for many air force aircraft and a design concern for future high-performance aircraft. High acoustic loads can produce cracks in thin, stiffened, aircraft skins or panels. The acoustic input is typically a band-limited random excitation. Fatigue occurs when there are lightly damped resonant bending modes of the skins/panels within the excitation bandwidth. The dynamic response becomes highly nonlinear for large response amplitudes.

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Early sonic fatigue prediction techniques were based on single-mode, linear response models of beams and plates [1]. The methods give reasonable results for simple structures at low acoustic levels, but are not accurate for complex structures with multi-mode behavior at high amplitudes. Finite element analysis is widely used for linear dynamic response prediction. However, if finite element analysis is used for prediction of the nonlinear, stochastic sonic fatigue response, the computation burden can be enormous for anything but the simplest of structures [2].

Some efforts have focused on reducing the nonlinear finite element model through modal transformation and truncation. The modal model is much smaller and the computational expense of the time integration is greatly reduced. The linear portions of the model come directly from a normal modes analysis of the finite element model. The nonlinear portions of the model are obtained through direct manipulation of the nonlinear stiffness [3,4] or from estimation schemes using static nonlinear finite element solution results [5–10]. Three recently reported methods [4,8,9] were used to predict the response of a clamped–clamped beam [9]. The results were compared with data from a well-characterized experiment. The methods were able to predict power spectral densities (PSDs) of both displacement and strain accurately. However, differences among the individual methods were not fully investigated. In this paper, the technical challenges of nonlinear modal modeling are reviewed. Then, an example problem is used to more deeply explore several nonlinear modal methods.

Some bounds on the problem are required to manage the ensuing discussion of the methods. First, elevated temperatures often accompany acoustic loading in the sonic fatigue problem. Elevated temperatures can change material properties, change the stress state, and buckle the structure. It will be assumed that the temperature environment is constant with time (but not necessarily spatially) and the structure is not buckled. That is not to say that thermal stresses are neglected by the methods, but rather the stress can be manifested in the parameters of the linear portions of modal. Second, although stresses and strains are ultimately required for sonic fatigue prediction, the discussions will focus on displacement predictions. Displacement is the primary variable; stress and strain are derived from it. It is acknowledged that there will be differences in stress calculation among the various methods, but that is beyond the scope of this effort. Ultimately, the sonic fatigue prediction problem requires methods that will work for built-up aircraft structures. Some of the methods discussed herein may not be suitable for complex structures. Although the example problem discussed below is a straight beam, the intent is not to find the best method for beams or even planar structures, but rather general structures.

2. Nonlinear modal models

Finite element models with a large number of degrees of freedom (dof) can be reduced to a low-order system of modal equations. The typical linear finite element model produces a system of equations

$$\mathbf{M}\ddot{\mathbf{w}} + \mathbf{K}\mathbf{w} = \mathbf{f}(t), \tag{1}$$

where $\mathbf{w}$ is the temporally varying vector of nodal displacements; $\mathbf{M}$ and $\mathbf{K}$ are the linear mass and stiffness matrices. The vector, $\mathbf{f}(t)$, represents the nodal force vector as a function of time. The linear equations of motion can be uncoupled by transforming the physical coordinates, $\mathbf{w}$, to modal coordinates, $\mathbf{p}$,

$$\mathbf{w}(t) = \sum_{i=1}^{N} \boldsymbol{\phi}_i p_i(t), \tag{2}$$

where N is the number of dof in the finite element model. The normal mode shapes (eigenvectors), $\boldsymbol{\phi}_i$, are the basis vectors in the modal expansion. The normal modes are computed from the eigensolution of the homogeneous form of Eq. (1). The solution also produces the natural frequencies, ω_i. The mode shapes are scaled so that

$$\boldsymbol{\phi}^{\mathrm{T}}\mathbf{M}\boldsymbol{\phi} = \mathbf{I}, \quad \boldsymbol{\phi}^{\mathrm{T}}\mathbf{K}\boldsymbol{\phi} = \bar{\mathbf{K}} = \mathrm{diag}(\omega_1^2, \omega_2^2, \ldots, \omega_N^2). \tag{3}$$

Typically only a few modes are retained in the expansion and the displacement vector is approximated.

For a linear problem, the uncoupled modal equations are expressed as

$$\ddot{p}_r + 2\zeta_r\omega_r\dot{p}_r + \omega_r^2 p_r = \boldsymbol{\phi}_r^{\mathrm{T}}\mathbf{f}(t). \tag{4}$$

A term with the modal damping factor, ζ_r, has been inserted into the equation at this point to model the energy dissipation. For a nonlinear problem, the equation for an arbitrary mode can be written generally as

$$\ddot{p}_r + 2\zeta_r\omega_r\dot{p}_r + \omega_r^2 p_r + \theta_r(p_1, p_2, \ldots, p_n) = \boldsymbol{\phi}_r^{\mathrm{T}}\mathbf{f}(t), \tag{5}$$

where θ_r represents the nonlinear part of the model. Note that only n modes have been retained in the modal expansion. The nonlinear modal equations differ from linear modal equations in that they are coupled through the nonlinear function. Typically for structural applications, the nonlinear function has quadratic and cubic terms. The most general form of the nonlinear function is

$$\theta_r = \sum_{i=1}^{n}\sum_{j=i}^{n} B_r(i,j)p_i p_j + \sum_{i=1}^{n}\sum_{j=i}^{n}\sum_{k=j}^{n} A_r(i,j,k)p_i p_j p_k, \tag{6}$$

where the quadratic coefficients, B_r, and the cubic coefficients, A_r, belong to the "rth" modal equation and have indices to denote which group of terms they multiply.

The low-order system of nonlinear modal equations can be solved efficiently by direct time integration. Time histories of the modal displacements result. Time histories of selected physical displacements are computed from the modal displacements. Element strains are computed from the physical displacements using strain–displacement equations. Strains can also be computed directly from the modal displacements using a derived nonlinear strain function [7,8].

3. Evaluation of nonlinear stiffness coefficients

Several procedures are possible for evaluating the nonlinear stiffness coefficients in Eq. (6). The evaluation can be performed directly by manipulation of the nonlinear finite element stiffness matrices or indirectly from a set of nonlinear static solutions with either enforced displacements or applied loads. The details of these procedures and their suitability for practical implementation are discussed in the following paragraphs.

3.1. Direct evaluation

One means of obtaining the nonlinear portion of the modal model is by directly manipulating the nonlinear stiffness matrix through modal transformation [3,4]. Here, this procedure will be termed *direct evaluation*. Direct evaluation begins with the nonlinear finite element equations of motion

$$\mathbf{M}\ddot{\mathbf{w}} + [\mathbf{K} + \mathbf{K1}(\mathbf{w}) + \mathbf{K2}(\mathbf{w}, \mathbf{w})]\mathbf{w} = \mathbf{f}(t), \tag{7}$$

where the quadratic nonlinear stiffness matrix, $\mathbf{K1}$, is a linear function of the nodal displacements and the cubic nonlinear stiffness matrix, $\mathbf{K2}$, is a quadratic function of the nodal displacements. Substituting the modal coordinate transformation from Eq. (2) into the nonlinear stiffness matrices results in expressions for the nonlinear stiffness matrices in terms of mode shapes and modal coordinates

$$\mathbf{K1}(\mathbf{w}) = \sum_{i=1}^{N} p_i \mathbf{K1}(\boldsymbol{\phi}_i), \quad \mathbf{K2}(\mathbf{w}, \mathbf{w}) = \sum_{i=1}^{N}\sum_{j=1}^{N} p_i p_j \mathbf{K2}(\boldsymbol{\phi}_i, \boldsymbol{\phi}_j). \tag{8}$$

Substituting Eqs. (2) and (8) into Eq. (7) and pre-multiplying all terms by $\boldsymbol{\phi}^{\mathrm{T}}$ results in the nonlinear modal equations of motion,

$$\ddot{\mathbf{p}} + \bar{\mathbf{C}}\dot{\mathbf{p}} + (\bar{\mathbf{K}} + \bar{\mathbf{K}}\mathbf{1} + \bar{\mathbf{K}}\mathbf{2})\mathbf{p} = \boldsymbol{\phi}^{\mathrm{T}}\mathbf{f}(t), \tag{9}$$

where

$$\bar{\mathbf{K}}\mathbf{1} = \boldsymbol{\phi}^{\mathrm{T}}\mathbf{K1}(\mathbf{w})\boldsymbol{\phi}, \quad \bar{\mathbf{K}}\mathbf{2} = \boldsymbol{\phi}^{\mathrm{T}}\mathbf{K2}(\mathbf{w}, \mathbf{w})\boldsymbol{\phi}. \tag{10}$$

Modal damping is inserted into the equations of motion at this point as $\bar{\mathbf{C}}$, a diagonal matrix of $2\zeta_i\omega_i$. Finally, the modal expansion is truncated by retaining only n modes in order to make the computation practical. The nonlinear stiffness terms in Eq. (10) can be grouped into the form given by Eq. (6). The direct evaluation procedure requires access to the assembly of nonlinear stiffness matrices. This access is not available in most commercial finite element codes, making this approach less attractive for practical implementation.

3.2. Indirect evaluation—enforced displacements

The nonlinear coefficients can alternatively be evaluated from static finite element results [5,6]. The procedure requires linear and nonlinear enforced displacement solutions using any suitable commercial finite element code. Here, this procedure will be termed the *enforced displacement* procedure.

For a static enforced displacement, $\mathbf{w}_c$, Eq. (7) becomes

$$\mathbf{K}\mathbf{w}_c + \Gamma(\mathbf{w}_c) = \mathbf{f}, \tag{11}$$

where $\Gamma(\mathbf{w}_c)$ represents the nonlinear terms evaluated at the enforced displacement. In the enforced displacement problem, the finite element method is used to obtain the constraint force, **f**, for the given enforced displacement. Note that for a linear analysis, the nonlinear terms are zero. Thus, the term $\Gamma(\mathbf{w}_c)$ can be numerically evaluated for a particular enforced displacement by

$$\Gamma(\mathbf{w}_c) = \mathbf{f}_T - \mathbf{K}\mathbf{w}_c = \mathbf{f}_T - \mathbf{f}_L, \tag{12}$$

where $\mathbf{f}_L$ represents the constraint force for the linear analysis and $\mathbf{f}_T$ represents the constraint force for the total problem, i.e. the nonlinear analysis problem.

The enforced displacement vector is chosen as a linear combination of the modal basis vectors. A series of cases are obtained from a set of enforced displacement vectors and a system of equations is developed from the cases. Finally, the system of equations is manipulated to extract the coefficients in Eq. (6). The nonlinear coefficients extraction is done external to the finite element code. Details of the procedure can be found in Refs. [5,6].

The enforced displacement procedure was originally developed to determine an interim model that could be used as part of an equivalent linearization scheme. The software written by the originators of the procedure is called ELSTEP [11]. In the present study, the enforced displacement procedure was implemented using NASTRAN and MATLAB without the use of the ELSTEP code, and the nonlinear model was used directly without any equivalent linearization procedure. The nonlinear modal model from the enforced displacement procedure is equivalent to that obtained by direct evaluation. The enforced displacement procedure is preferred over the direct evaluation procedure because it can be implemented using any suitable, commercial, finite element code.

An alternative enforced displacement solution has been proposed for the elevated temperature case [10]. Thermal stresses may not be adequately included in static linear solutions. A second set of nonlinear solutions is required instead of the linear solution. Also, the extraction of the nonlinear stiffness terms is slightly different in the alternative solution. However, the solution remains an enforced displacement procedure requiring linear combinations of the modal basis vectors.

3.3. Indirect evaluation—applied loads

Another means of determining the nonlinear terms is through estimation using static applied loads solutions. Like the enforced displacement procedure, access to the internal workings of the finite element code is not required. Implementation of the estimation procedure is external to the finite element code. This procedure differs from the enforced displacement procedure in that the static solutions are applied load solutions. Fittingly, the procedure will be called the *applied loads* procedure.

In the applied loads procedure, nonlinear static solutions are found for multiple load cases using any suitable finite element code. The load cases are selected to exercise the nonlinear effects and produce displacements with the proper modal components. Applied load vectors that are comprised of appropriately scaled linear combinations of the modal basis vectors are an obvious choice [8]. The displacements are obtained and then projected onto the modal space to obtain modal amplitudes for the given load. Enough cases are run so that a system of equations is developed. Coefficients of the nonlinear model are estimated from the system of equations on a mode-by-mode basis.

At first glance, the applied loads procedure appears to be similar to the enforced displacement procedure. There is a difference, however. The nonlinear static displacements computed from applied loads contain both bending and membrane components due to the nonlinear stiffness of the finite element formulation. Membrane displacements occur naturally without explicitly including applied loads based upon membrane modes. In the enforced displacement procedure, membrane displacements must be included explicitly by including membrane modes in the set of enforced displacements. Details of the applied loads procedure can be found in Ref. [8].

4. Modal basis vectors and membrane effects

The number and type of modes retained in the basis set have a significant effect on response prediction. Typically for beams, plates, and panels, low-order modes involving bending (out-of-plane) displacement are of primary importance in the response. However, as a thin skin undergoes large displacement, it also stretches. The geometric nonlinearity couples bending displacements with membrane (in-plane) displacements through quadratic stiffness terms. Therefore, the modal basis set must include independent membrane displacement vectors to accommodate the membrane displacements.

Independent membrane basis vectors may not exist for the case of non-planar or composite structures with asymmetric laminates. In these cases, the low-frequency mode shapes that are directly excited may contain both bending and membrane displacements. However, additional basis vectors may be required to capture the membrane effects. These additional vectors are not directly excited. They are indirectly excited though nonlinear coupling.

Several approaches for selecting membrane basis vectors are discussed in the following paragraphs. Approaches for including the effects of membrane displacements in the bending displacements via static condensation are also discussed.

4.1. Neglecting membrane effects

The first option to modeling the membrane displacements is simply to neglect them. Only the low-frequency bending modes in the excitation bandwidth are retained in the basis set. The nonlinear modal model will have cubic stiffness terms that are too large, resulting in an overly stiff model. This happens because the quadratic nonlinear terms in Eq. (6), which tend to reduce the total nonlinear stiffness, are dependant on membrane displacements.

4.2. Normal membrane modes

The second option is to model the membrane displacement by including normal membrane modes in the basis set. For straight beams or planar structures, these are normal modes that involve only in-plane motion and are much higher in frequency than the bending modes included in the model. Although normal membrane modes are relatively easily to find for a planar structure, a decision on which and how many membrane modes to include is difficult. They are not directly excited by the external forcing, but are only excited indirectly through quasi-static, nonlinear coupling with bending modes. As the structure becomes more complex or non-planar, normal modes may contain both membrane and bending displacements, making it very difficult to identify a membrane basis set.

The resulting nonlinear model may have high-frequency modes in the basis set. The high-frequency modes can cause integration instability if the time step of integration is too large. The stability concerns and computational burden can be alleviated through modal condensation, which will be explained in a following section.

4.3. Companion membrane modes

An alternate strategy for the modeling of membrane effects is to synthesize the necessary membrane displacement vectors [9,10]. The synthesized modes have been termed *companion* membrane [9] or *dual* [10] modes. A companion membrane mode is generated for each bending mode to provide a basis vector for the membrane stretching induced by the large displacement nonlinearity.

The companion membrane mode is generated for each bending mode in the model using a nonlinear finite element static solution. The linear force due to the enforced displacement of a selected bending mode is found. This force, $\mathbf{K}\boldsymbol{\phi}_b$, is scaled and applied to the finite element model in a nonlinear static solution. A nonlinear deflected shape, $\boldsymbol{\phi}_{nl}$, containing both bending and induced membrane displacement, is then computed from

$$[\mathbf{K} + \mathbf{K1}(\boldsymbol{\phi}_{nl}) + \mathbf{K2}(\boldsymbol{\phi}_{nl}, \boldsymbol{\phi}_{nl})]\boldsymbol{\phi}_{\mathrm{nl}} = \alpha\mathbf{K}\boldsymbol{\phi}_b, \tag{13}$$

where α is a scalar. The bending mode's projection on the nonlinear deflected shapes is removed through the Gram–Schmidt procedure. The result is a companion membrane mode. The procedure presented in Ref. [9] is modified to that presented here so that standard commercial finite element code can be used to generate the nonlinear deflected shape. Ref. [10] presents a procedure to compute a *dual* mode which is directly analogous to the procedure just described. The primary difference is the process used to remove the bending mode's projection from the nonlinear deflected shape.

The companion membrane modes generated by this process are representations of the nonlinear membrane displacements resulting from large-amplitude response in a particular bending mode. For all but the simplest structures, a companion membrane mode will not be a normal mode of the structure. Instead, it will be a linear combination of several normal modes. If more than one companion membrane mode is used in the nonlinear modal model, the companion membrane modes may not be orthogonal to each other. The lack of orthogonality is not a concern if the companion membrane modal displacements are condensed into the bending modal displacements as discussed later.

4.4. Explicit physical condensation of membrane dof

A fourth approach includes the effects of membrane stretching in the nonlinear modal model through static condensation of the membrane dof into the bending dof in the physical domain [4]. The approach is applicable to models whose global displacement vector can be partitioned into separate bending dof and membrane dof subsets. The approach is applicable to most planar structures and some non-planar structures that can be modeled with curvilinear coordinates such that the global displacements can be partitioned.

The membrane inertia is neglected and the nonlinear and linear stiffness of the membrane dof are statically condensed in a process analogous to the Guyan reduction. Since the condensation requires access to the assembly of the stiffness matrix, direct evaluation must be used to obtain the nonlinear modal model. Only bending modes are present in the modal model, but the effects of the membrane dof are seen in the nonlinear coefficients. This approach requires a specialized finite element code to implement. This fact, coupled with the applicability to a limited range of problems, makes it less attractive for practical application.

4.5. Explicit modal condensation of membrane effects

Modal condensation, as employed here, is simply expressing the membrane modal displacements in terms of the bending modal displacements. This condensation in the modal domain is analogous to static condensation in the physical domain. The first step in the process is to partition the modal equations into bending and membrane equations. The membrane modes are not directly forced and respond quasi-statically. Neglecting the acceleration and velocity terms, the membrane equations become

$$\bar{\mathbf{K}}_m \mathbf{p}_m + \boldsymbol{\theta}_m(p_1, p_2, \ldots, p_n) = 0, \tag{14}$$

where

$$\bar{\mathbf{K}}_m = \mathrm{diag}(\omega_{m1}^2, \omega_{m2}^2, \ldots, \omega_{mq}^2), \tag{15}$$

and where the subscripts $m1$–mq denote indices for the membrane modes. For the membrane modes, the only significant entries in the nonlinear function are the quadratic terms involving the bending modes. Eq. (14) can therefore be solved for the membrane modal amplitudes and the expressions substituted into the bending equations. For example, for a case with two bending modes and three normal membrane modes, the solution is

$$\begin{Bmatrix} p_3 \\ p_4 \\ p_5 \end{Bmatrix} = -\bar{\mathbf{K}}_m^{-1} \begin{bmatrix} B_3(1,1) & B_3(2,2) & B_3(1,2) \\ B_4(1,1) & B_4(2,2) & B_4(1,2) \\ B_5(1,1) & B_5(2,2) & B_5(1,2) \end{bmatrix} \begin{Bmatrix} p_1^2 \\ p_2^2 \\ p_1 p_2 \end{Bmatrix}, \tag{16}$$

where the subscripts 1 and 2 denote the bending modes, and the subscripts 3, 4, and 5 denote the membrane modes. When substituted in the bending equations, the expressions for the membrane modal amplitudes convert the quadratic terms involving both bending and membrane modes to cubic terms involving one or more of the bending modes. The result is a nonlinear model with only bending modes where the resulting cubic terms have been softened. This condensation approach will be referred to as *general condensation*.

The condensation of companion membrane modes is much simpler than the general condensation of normal membrane modes. The condensation process is restricted so that a companion membrane mode can only affect the bending mode from which it was synthesized. The condensation for a bending and membrane pair is described in Refs. [3,8]. The modal equations for the pair are

$$\ddot{p}_b + 2\zeta_b\omega_b\dot{p}_b + \omega_b^2 p_b + A_b(b,b,b)p_b^3 + B_b(b,m)p_b p_m + \cdots = \boldsymbol{\phi}_r^{\mathrm{T}}\mathbf{f}(t), \tag{17}$$

$$k_m p_m + B_m(b,b)p_b^2 = 0, \tag{18}$$

where the subscripts b and m denote the bending and membrane modes, respectively. Note that Eq. (17) is Eq. (5) with some of the nonlinear terms expressed. Eq. (18) is the single mode version of Eq. (14). The assumption is that the in-plane inertia of the structure is negligible so that the companion membrane mode responds in a quasi-static manner. The companion membrane modal amplitude in terms of the amplitude of the bending mode is found from Eq. (18) as

$$p_m = -(B_m(b,b)/k_m)p_b^2. \tag{19}$$

Substituting Eq. (19) into Eq. (17) produces

$$\ddot{p}_b + 2\zeta_b\omega_b\dot{p}_b + \omega_b^2 p_b + (A_b(b,b,b) - B_b(b,m)B_m(b,b)/k_m)p_b^3 + \cdots = \boldsymbol{\phi}_r^{\mathrm{T}}\mathbf{f}(t), \tag{20}$$

which converts the quadratic term in Eq. (17) into a cubic term. This clearly demonstrates that the effect of the companion membrane mode is to soften the cubic term in the bending mode.

This condensation approach will be referred to as *simple condensation*. In simple condensation, each companion mode will soften a single term in its bending companion. The softened term is the primary cubic nonlinear term in the bending equation and does not involve coupling to another bending mode. In contrast, general condensation can affect cross cubic terms, i.e. terms that involve more than one bending mode.

4.6. Implicit condensation of membrane effects

The effects of membrane displacements can be implicitly included in the model through the estimation of the nonlinear stiffness coefficients. The approach [7,8] uses the applied loads procedure to estimate the coefficients and will be referred to as *implicit condensation*. The approach restricts the nonlinear function to cubic stiffness terms and considers only bending modes. Terms involving coupling among three modes are also neglected. The nonlinear function with these restrictions can be written as

$$\theta_r = \sum_{i=1}^{n} A_r(i,i,i)p_i^3 + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\{A_r(i,i,j)p_i^2 p_j + A_r(i,j,j)p_i p_j^2\}. \tag{21}$$

The membrane modes are usually coupled to the bending modes through quadratic terms. These terms are not included in the nonlinear function, nor are membrane modes explicitly included in the basis set. Yet, the values of the individual bending deflections in the static solution set are affected by the membrane stiffness. The applied loads procedure does not constrain the structure from stretching as it is loaded. The nonlinear effects of the membrane stretching are exercised in the finite element code and can be seen in the static solution. When the model coefficients are estimated, the membrane stretching softens the cubic coefficients of the bending mode. In this manner, the method implicitly condenses the effect of the membrane modes into the bending modes. No overt effort to model the membrane stretching is required.

The number of estimated coefficients can vary with the implicit condensation method. All the allowable terms are given in Eq. (21). Regression schemes are proposed [8] that vary which of the allowable terms are used. The goal of the regression schemes is to build a model with the least number of nonlinear terms. However, the final model from the regression schemes can be sensitive to the particular static solution set, and in some cases may omit important terms. The regression schemes are therefore not recommended. In this paper, all of the allowable terms are retained.

There are other important issues with the implicit method such as the selection of the applied load set and the computation of strains [8,9]. In short, the load set must exercise all the appropriate nonlinear effects. Here, the load set is chosen as linear combinations of the scaled bending mode shapes [9]. Membrane strains cannot be computed directly because membrane displacements are not computed. Instead, strains are computed directly from the modal displacements with an estimated nonlinear function. The details are discussed in Refs. [8,9].

5. Methods for nonlinear modal modeling

Five modeling methods were selected for comparison with each other and with analytical predictions for a clamped–clamped beam example problem [9,12]. The first method, referred to as the *bending modes* method, includes only bending modes in the nonlinear modal model. This method is based on the approach described in Ref. [11] and uses the enforced displacement procedure to evaluate nonlinear stiffness coefficients.

The second method is the *bending and membrane modes* method. This method uses bending and normal membrane modes to compute the nonlinear coefficients. The enforced displacement procedure is used to compute the nonlinear coefficients. Generalized condensation is then used to incorporate the membrane modes into the bending equations.

The third method is the *companion modes* method. It uses pairs of bending and companion membrane modes in the basis set. The enforced displacement procedure is used to compute the nonlinear coefficients. Simple condensation is then used to express each companion membrane mode in terms of its bending mode twin. This is the method reported in Ref. [9].

The fourth method is referred to as the *physical condensation* method. It is the method of Shi and Mei [4]. Physical condensation is used to express the membrane dof in terms of the bending dof. Since physical condensation requires access to the assembly of the stiffness matrix, direct evaluation must be used to obtain the nonlinear modal model.

The fifth and final method evaluated will be referred to as the *implicit condensation* method. This method uses the applied loads procedure to evaluate the nonlinear coefficients and thus uses only bending modes in the modal basis. This is the method of McEwan [7,8], with the exception that all nonlinear terms are included in the model instead of the regression scheme to selectively eliminate insignificant terms.

6. The example problem

A clamped–clamped beam example problem was used to evaluate the accuracy of the modeling methods described in the previous sections. The example problem was based on the experiment reported in Ref. [12]. The test specimen was a thin steel beam clamped at both ends to a rigid fixture to approximate fixed end conditions. Test and analysis revealed that the boundary conditions were not ideally fixed, so they were modeled with stiff spring elements. The objective of the present study is to compare several nonlinear modal methods against each other and published analytical results [13]. Since the analytical results are for a beam with perfectly fixed ends, the example beam problem was also modeled with ideal clamped ends.

The beam used in the example problem was 9.0 in long, 0.5 in wide, and 0.031 in thick. The beam material was high-carbon steel with a Young's modulus of 29.7 Mpsi, a shear modulus of 11.6 Mpsi, and a mass density of $7.36 \times 10^{-4}\,\mathrm{lb\,s^2/in^4}$. The beam was free from preload and was subjected to a random base motion. The input was Gaussian with a flat, band-limited (0–500 Hz) spectrum. Root mean square (rms) input acceleration levels of 0.5, 1, 2, 4 and 8 g were used in the simulations. The input spectrum excited the first two symmetric bending modes of the beam. These two modes occurred at 79.0 and 427 Hz in the linear finite element model. Modal damping ratios of 0.003 and 0.005 were assumed for these two modes, respectively.

A 2-D finite element model with 20, 2-node beam elements was used to model the example problem. Only half of the beam was modeled, using symmetry, since the inertial loading could only excite symmetric bending modes. NASTRAN was used to compute the necessary solutions for all the methods except the physical condensation method. This method used an in-house code implemented in MATLAB. The first six normal membrane modes were used in the bending and membrane modes method. These modes had frequencies of 22.3, 44.8, 67.6, 90.8, 114, and 139 kHz. A companion membrane mode was generated for each of the two bending modes in the companion modes method.

6.1. One-mode models

The candidate methods were first used to form one-mode models of the example problem. The first bending mode was used in the models since it dominates the response of the beam to base excitation. A one-mode model was used so that nonlinear stiffness coefficients could be compared to published analytical results [13]. However, the one-mode models were not as accurate in predicting response as the two-mode models discussed later. The companion membrane mode or normal membrane modes were condensed into the one-mode model according to the particular method.

A one-mode version of the nonlinear modal model expressed in Eqs. (5) and (6) is given by

$$\ddot{p}_1 + 2\zeta_1\omega_1\dot{p}_1 + \omega_1^2 p_1 + A_1(1,1,1)p_1^3 = \boldsymbol{\phi}_1^{\mathrm{T}}\mathbf{f}(t). \tag{22}$$

This model can be converted to an equivalent model in physical coordinates using the mode shape. This is done to compare the models obtained from the candidate methods to the one derived from the literature. By selecting the out-of-plane displacement at the center of the beam, x_c, as the physical coordinate of interest, the model becomes

$$\ddot{x}_c + 2\zeta_1\omega_1\dot{x}_c + \omega_1^2 x_c + \tilde{A}_1(1,1,1)x_c^3 = \boldsymbol{\phi}_{1c}\boldsymbol{\phi}_1^{\mathrm{T}}\mathbf{f}(t), \tag{23}$$

where $\boldsymbol{\phi}_{1c}$ denotes the row entry of the first mode shape vector that corresponds to the center dof. The cubic coefficient is expressed in physical units and is denoted as $\tilde{A}_1(1,1,1)$.

An analytical value for the cubic nonlinear coefficient was determined by curve-fitting data found in Ref. [13] for frequency ratios at several amplitude ratios. The curve fit procedure found in Ref. [14] was used to estimate the coefficient from an approximation to the nonlinear backbone curve. The resulting expression for the cubic coefficient is

$$\tilde{A}_1(1,1,1) = (0.0585)\frac{\omega_1^2}{r^2}, \tag{24}$$

where r is the radius of gyration and ω_1 is the first linear natural frequency. The cubic coefficient was determined and is presented in Table 1.

The cubic coefficients from the candidate nonlinear modal methods are also presented in the table in physical units. All the methods except the bending modes method produce cubic coefficients that agree well with the analytical value. The cubic coefficient from the bending modes method is 50% higher than the analytical value. That is, all the methods, except the bending modes methods, have softened the cubic coefficient. The impact of the difference in cubic coefficients on the rms displacement response at the center of the beam is shown in Table 1. All the methods except for the bending modes method yielded values of 0.047 or 0.048 in. The value from the bending modes method was somewhat lower at 0.044 in. The rms displacements were estimated from simulations with 10 million time points integrated at 20 kHz for each method with an 8 g rms base motion input.

The PSDs of beam center displacement were estimated for each of the methods. The PSDs for the implicit condensation and the bending modes method are shown in Fig. 1. The PSD of the predicted response for the other three methods are essentially the same as that for the implicit condensation and are omitted from the figure for clarity. The results shown in Table 1 and Fig. 1 show that the higher value for the cubic coefficient obtained when the membrane stretching is neglected (the bending modes method) results in: (1) a lower rms value, (2) a lower peak in the PSD of the response, (3) a peak that is shifted higher in frequency, and (4) a broader peak.

6.2. Two-mode models

Traditionally, a single-mode model is thought to be adequate to model the response of most stiffened-skin aircraft structures, since they are assumed to respond predominantly in their first mode. However, a multi-mode nonlinear model has terms that couple the first mode to higher

Table 1
Single-mode models from various methods

Method	$\tilde{A}_1(1,1,1)$ (in^{-2} s^{-2})	Displacement[a] (in)
Analytical	1.80×10^8	—
Bending modes	2.70×10^8	0.044
Bending and membrane modes	1.84×10^8	0.048
Companion modes	1.84×10^8	0.048
Physical condensation	1.83×10^8	0.047
Implicit condensation	1.79×10^8	0.047

[a]Rms displacement of the center of the beam for a simulated 8 g, 0–500 Hz random load.

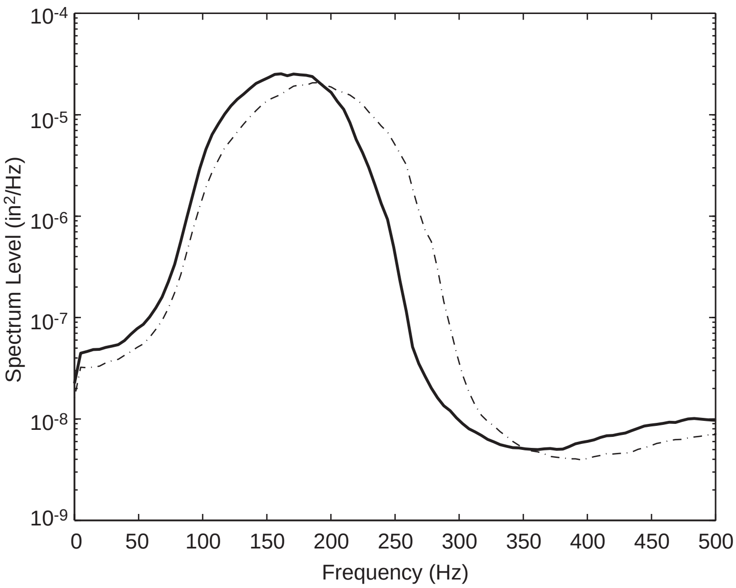

Fig. 1. The PSD of the response of the center of the beam due to an 8 g, 0–500 Hz base excitation using single-mode models: ——, the implicit condensation method; –·–, the bending modes method.

modes in the model. In the clamped–clamped beam example problem, nonlinear coupling terms can cause response in the second mode to affect the first mode and vice versa. It will be shown in this section that the nonlinear coupling terms in a two-mode model will affect the response in the first mode. It will also be shown that the way in which the membrane stretching is modeled affects the magnitude of the coupling terms and thus affects the predicted response.

A two-mode nonlinear model has the form

$$\ddot{p}_1 + 2\zeta_1\omega_1\dot{p}_1 + \omega_1^2 p_1 + A_1(1,1,1)p_1^3 + A_1(1,1,2)p_1^2 p_2 + A_1(1,2,2)p_1 p_2^2 + A_1(2,2,2)p_2^3 = f_1(t), \tag{25}$$

$$\ddot{p}_2 + 2\zeta_2\omega_2\dot{p}_2 + \omega_2^2 p_2 + A_2(1,1,1)p_1^3 + A_2(1,1,2)p_1^2 p_2 + A_2(1,2,2)p_1 p_2^2 + A_2(2,2,2)p_2^3 = f_2(t), \tag{26}$$

where f_1 and f_2 are modal force components. Note that there are no quadratic nonlinear terms since they have been condensed into the cubic terms. The modal equations are transformed to physical coordinates by selecting the out-of-plane displacement at the center of the beam as the coordinate of interest. The two-mode nonlinear coefficients for the five modeling methods were computed and converted to physical units. The coefficients are listed in Table 2.

The $\tilde{A}_1(1,1,1)$ coefficient for the bending modes method, shown in Table 2, is still 50% larger than the $\tilde{A}_1(1,1,1)$ coefficients from the other methods. The $\tilde{A}_1(1,1,1)$ coefficients for the two-mode models all agree with the corresponding single-mode cubic terms in Table 1.

Table 2
Nonlinear coefficients from the two-mode models

Coefficient[a] ($in^{-2}\,s^{-2}$)	Method				
	Bending modes	Bending & membrane modes	Companion modes	Physical condensation	Implicit condensation
$\tilde{A}_1(1,1,1)$	2.70×10^{8}	1.84×10^{8}	1.84×10^{8}	1.83×10^{8}	1.82×10^{8}
$\tilde{A}_1(1,1,2)$	6.41×10^{8}	4.92×10^{8}	6.41×10^{8}	4.90×10^{8}	4.79×10^{8}
$\tilde{A}_1(1,2,2)$	5.56×10^{9}	2.17×10^{9}	5.56×10^{9}	2.17×10^{9}	2.16×10^{9}
$\tilde{A}_1(2,2,2)$	4.31×10^{9}	1.68×10^{9}	4.31×10^{9}	1.68×10^{9}	1.36×10^{9}
$\tilde{A}_2(1,1,1)$	1.68×10^{8}	1.29×10^{8}	1.68×10^{8}	1.23×10^{8}	1.26×10^{8}
$\tilde{A}_2(1,1,2)$	4.36×10^{9}	1.70×10^{9}	4.36×10^{9}	1.63×10^{9}	1.67×10^{9}
$\tilde{A}_2(1,2,2)$	1.01×10^{10}	3.94×10^{9}	1.01×10^{10}	3.79×10^{9}	3.66×10^{9}
$\tilde{A}_2(2,2,2)$	2.44×10^{10}	1.50×10^{10}	1.50×10^{10}	1.45×10^{10}	9.03×10^{9}

[a]The coefficients are from nonlinear modal models written in physical coordinates.

The coefficients for the companion modes method are identical to those from the bending modes method, except for two values. Simple condensation restricted the condensation of the companion membrane modes from affecting the cross-coupling terms. Thus, only the primary cubic terms, $\tilde{A}_1(1,1,1)$ and $\tilde{A}_2(2,2,2)$, were softened by the companion membrane modes.

All the coefficients of the bending and membrane modes method were affected by general condensation. Before condensation of the membrane modes, the nonlinear coefficients of the bending modes are exactly the same as those from the bending modes method. After condensation, all of the coefficients are different from the bending modes method. Only the primary cubic terms, $\tilde{A}_1(1,1,1)$ and $\tilde{A}_2(2,2,2)$, from the bending and membrane modes method agree with the companion modes method. Thus, the softening of the primary nonlinear terms is the same for the two methods.

The coefficients from the physical condensation, implicit condensation, and the bending and membrane modes methods are all very similar. These three methods use very different means of modeling the membrane stretching and very different means of extracting the nonlinear coefficients. Yet, the coefficients from the physical condensation method vary only slightly from the coefficients from the bending and membrane modes methods. The coefficients from the implicit condensation method vary a bit more; only the $\tilde{A}_2(2,2,2)$ term is significantly different from the other two methods.

Simulations were run using the two-mode models. Response was calculated for the same input as the single-mode model (a zero-mean, normal, 8 g rms, 0–500 Hz band-limited, base motion input). Again 10 million time points integrated at 20 kHz were produced for each method. The PSD of the beam center displacement is shown in Fig. 2 for the implicit condensation, companion modes, and bending modes methods. The PSDs from the physical condensation method and the bending and membrane modes method are omitted from the figure since they are essentially the same as the PSD plot from the implicit condensation method, as one would expect from the

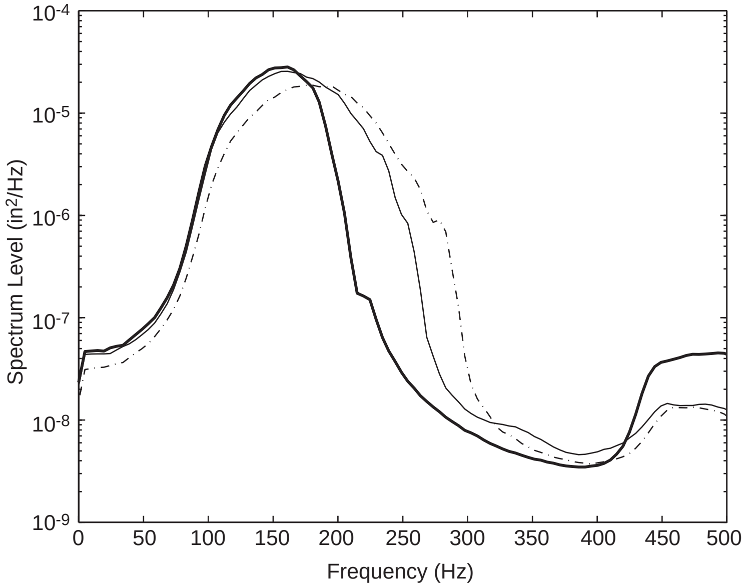

Fig. 2. The PSD of the response of the center of the beam due to an 8 g, 0–500 Hz base excitation using two-mode models: ——, the implicit condensation method; ——, the companion modes method; –·–, the bending modes method.

similarity of their model coefficients. The PSDs for the bending modes method and companion modes method are very similar to their respective single-mode PSD plots shown in Fig. 1, with variations occurring only at the higher frequencies. Thus, the two-mode models for the bending modes method and companion modes methods simply added response for the second mode when compared to the one-mode models. Little coupling between the two modes occurs with these models. However, with the implicit condensation, physical condensation, and the bending and membrane modes methods, coupling between the two modes did occur. The PSD from the implicit condensation method is significantly different from the PSDs of the bending modes method and the companion modes method. The first-mode peak is significantly narrower and the amplitude of the second mode is raised in the PSD for the implicit condensation method. This indicates that vibration energy from the first mode has been transferred to the second mode.

The rms displacements for the 8 g load case simulations are given in Table 3. The only value that is significantly different from the others is the value for the companion modes method. The close agreement between the rms values for the bending modes method and the other methods masks the fact that their PSDs are significantly different.

Rms displacements for simulations with 2 g loading are also listed in Table 3. At this reduced input level, the rms displacements for each of the methods are now nearly equal. Fig. 3 shows the PSDs for the 2 g simulation cases. Note that, again, the PSDs for the physical condensation method and the bending and membrane modes method agree closely with the implicit condensation method and are not shown for the sake of clarity. Differences in the PSD plots can be seen, but they are much smaller than for the 8 g load case.

Table 3
The rms displacement of the center of the beam from simulations

Case	Method				
	Bending modes (in)	Bending & membrane modes (in)	Companion modes (in)	Physical condensation (in)	Implicit condensation (in)
8 g loading ($\zeta_1 = 0.3\%$, $\zeta_2 = 0.5\%$)	0.043	0.043	0.046	0.042	0.042
2 g loading ($\zeta_1 = 0.3\%$, $\zeta_2 = 0.5\%$)	0.019	0.020	0.020	0.020	0.020
8 g loading ($\zeta_1 = 1\%$, $\zeta_2 = 1\%$)	0.031	0.033	0.032	0.033	0.032

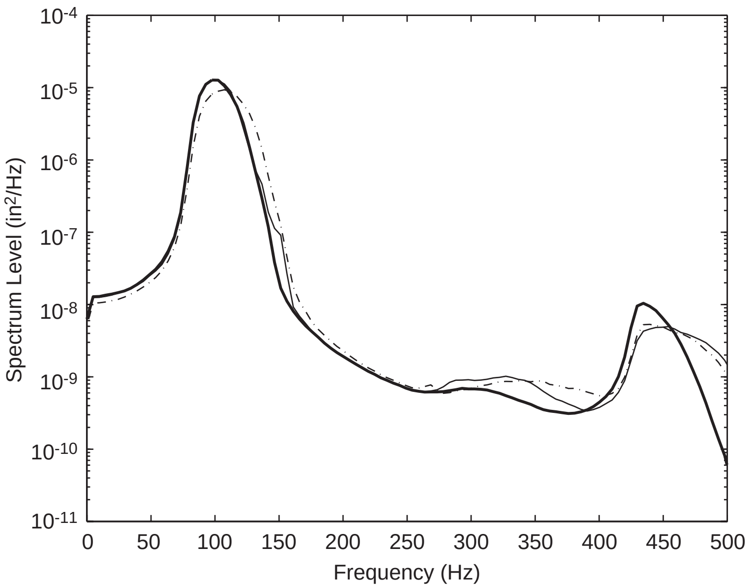

Fig. 3. The PSD of the response of the center of the beam due to a 2 g, 0–500 Hz base excitation using two-mode models: ——, the implicit condensation method; ——, the companion modes method; – · –, the bending modes method.

Rms displacements for simulations with 8 g loading and modal damping ratios for both modes increased to 0.01 are also listed in Table 3. The increased damping has reduced the rms displacements compared to the nominal damping case. In addition, the value for the companion modes method is nearly equal to the values for the other methods. The PSDs for the models with increased damping are shown in Fig. 4. The added damping has reduced the differences among the curves.

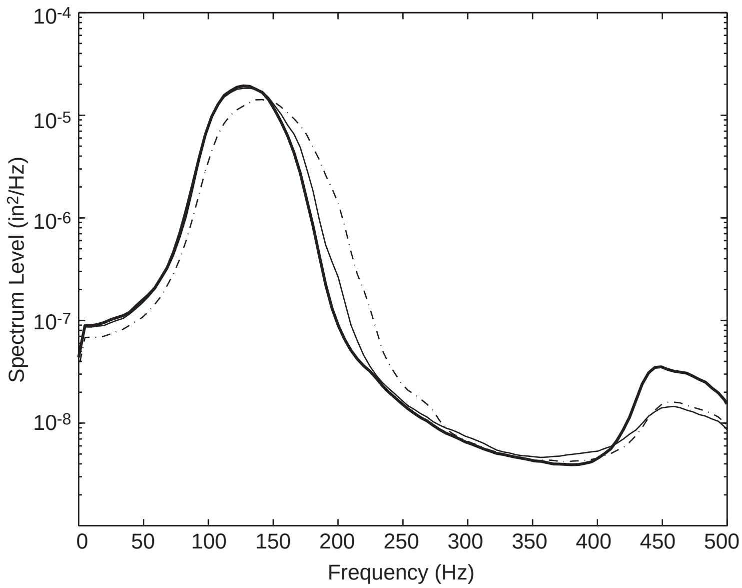

Fig. 4. The PSD of the response of the center of the beam due to an 8 g, 0–500 Hz base excitation using two-mode models. The damping for the two modes is increased to 1% of critical for this plot. ——, the implicit condensation method; ——, the companion modes method; –·–, the bending modes method.

7. Conclusions

Accurate response prediction is important in designing aircraft structures for sonic fatigue. Recently, methods that reduce a nonlinear finite element model to a low order system of nonlinear modal equations have been proposed. Accurate response prediction can be accomplished through numerical time integration of the nonlinear modal model and the computational burden is greatly reduced compared to integration of a full-order model. Five methods for obtaining the nonlinear modal model were summarized. The methods were compared using a clamped–clamped beam as an example problem.

As a constrained beam or plate-like structure undergoes large-amplitude displacements, it also stretches. The primary difference between the methods reviewed is the manner in which the stretching is modeled. If stretching is neglected, as is done in the bending modes method, the resulting modal model will have nonlinear terms that are too large. This method is the simplest to implement and retains only the low-frequency modes of interest. The bending modes method may provide a reasonable approximation, but a simulation with single-mode models did show a significant departure in the frequency content of the displacement PSD.

The companion modes method synthesizes a membrane mode to accompany each of the low-frequency bending modes in order to model the stretching. The companion modes can then be condensed into the bending modes. The condensation softens some of the cubic terms in the nonlinear model. The model from this method is more accurate than the models from the bending modes method. In the absence of interaction between multiple modes, the companion modes method appears to provide an acceptable approximation. However, this method showed significant differences from the other methods for the two-mode models, where nonlinear modal coupling was significant.

The implicit condensation method, the physical condensation method, and the bending and membrane modes methods allow softening of the cross terms in the nonlinear modal model. The modeling of structural stretching is done completely differently among these three methods, yet they resulted in similar models. The simulation results for a two-mode system showed the importance of the cross term softening. For large displacements, energy can be transferred from the primary mode to higher modes. This energy transfer is governed by the nonlinear cross terms.

The physical condensation method handles the membrane stretching through static condensation of the physical membrane dof into the bending dof. The approach is applicable to models whose global displacement vector can be partitioned into separate bending dof and membrane dof subsets. This includes most planar structures and some non-planar structures that can be modeled with curvilinear coordinates such that the global displacements can be partitioned. In addition, use of this method requires access to the assembly of the nonlinear stiffness matrices and therefore requires specialized finite element code. The method is very useful as a research tool, but is not practical as a design tool for complex structures.

The bending and membrane modes method accounts for the stretching displacements by including normal membrane modes in the basis set. For beam or planar structures, these modes can be easily selected. As the structure becomes more complex, however, it becomes practically impossible to identify how many and which modes to include. Once found, membrane modes are easily condensed into the bending modes. Although the bending and membrane modes method is feasible on a complex structure, it becomes too difficult to implement.

In the implicit condensation method, the effects of the stretching are implicitly included in the nonlinear model when the nonlinear coefficients are estimated. The method uses nonlinear static finite element solutions from applied loading cases. The stretching is not explicitly included in the nonlinear model, but it affects the values of the deflections in the static solution set. When the model coefficients are estimated, the nonlinear effect of the stretching softens the cubic coefficients of the low-frequency modes that are retained. In this manner, the method implicitly condenses the effect of the stretching into the low-frequency bending modes.

Of the five methods studied, the implicit condensation method appears to be the most practical. The method softens the cross terms correctly, yet, other than the bending modes method, it is the easiest to implement. It provides an accurate nonlinear model without the need to include membrane basis vectors to model the effects of stretching. The method requires static solutions for a number of loading cases from any suitable commercial finite element software package. No access to the internal code is required. The coefficient estimation procedure can be implemented external to the commercial software.

Modal damping had a significant effect on the agreement of the response predictions from the nonlinear modeling methods investigated. The relatively light damping in the example problem ($\zeta_1 = 0.003$ and $\zeta_2 = 0.005$ for the first and second modes, respectively) resulted in significant differences in the PSDs and rms values among some of the methods. However, these differences became less apparent when the modal damping ratios were increased to $\zeta_1 = \zeta_2 = 0.10$. Thus, it is important to include realistic damping values in any design models.

While the methods presented in this work are very promising for predicting the response of beams to in-phase loading, additional work is needed. The effects of multiple dimensions (plates vs. beams), curvature, closely spaced modes, and traveling wave excitation need to be studied. Plates have higher modal density than beams. Curvature adds linear coupling between bending and membrane displacements in addition to the nonlinear coupling. Multi-bay stiffened structures can exhibit families of closely spaced or repeated modes. And, finally, traveling wave excitation will excite anti-symmetric modes in addition to symmetric modes.

References

[1] F. Rudder, H. Plumblee, Sonic fatigue design guide for military aircraft, *US Air Force Technical Report: AFFDL-TR-74-112*, 1975.
[2] P. Green, A. Killey, Time domain dynamic finite element modelling in acoustic fatigue design, *Proceedings of the Sixth International Conference on Recent Advances in Structural Dynamics*, University of Southampton, 1997, pp. 1007–1025.
[3] M. Nash, Nonlinear structural dynamics by finite element modal synthesis, *Ph.D. Dissertation*, Imperial College, The University of London, 1977.
[4] Y. Shi, C. Mei, A finite element time domain modal formulation for large amplitude free vibrations of beams and plates, *Journal of Sound and Vibration* 193 (1996) 453–464.
[5] S.A. Rizzi, A.A. Muravyov, Equivalent linearization analysis of geometrically nonlinear random vibrations using commercial finite element codes, *NASA/TP-2002-211761*, 2002.
[6] A.A. Muravyov, S.A. Rizzi, Determination of nonlinear stiffness with application to random vibration of geometrically nonlinear structures, *Computers and Structures* 81 (2003) 1513–1523.
[7] M.I. McEwan, J.R. Wright, J.E. Cooper, A.Y.T. Leung, A finite element/modal technique for nonlinear plate and stiffened panel response prediction, *AIAA-2001-1595*, 2001.
[8] M.I. McEwan, A combined modal/finite element technique for the non-linear dynamic simulation of aerospace structures, *Ph.D. Dissertation*, University of Manchester, England, 2001.
[9] J.J. Hollkamp, R.W. Gordon, S.M. Spottswood, Nonlinear sonic fatigue response prediction from finite element modal models: a comparison with experiments, *AIAA-2003-1709*, 2003.
[10] M.P. Mignolet, A.G. Radu, Validation of reduced order modeling for the prediction of the response and fatigue life of panels subjected to thermo-acoustic effects, *Proceedings of the Eighth International Conference on Recent Advances in Structural Dynamics*, University of Southampton, 2003.
[11] S.A. Rizzi, A.A. Muravyov, Improved equivalent linearization implementations using nonlinear stiffness evaluation, *NASA/TM-2001-210838*, 2001.
[12] R.W. Gordon, J.J. Hollkamp, S.M. Spottswood, Nonlinear response of a clamped–clamped beam to random base excitation, *Proceedings of the Eighth International Conference on Recent Advances in Structural Dynamics*, University of Southampton, 2003.
[13] G. Singh, G.V. Rao, N.G.R. Iyengar, Re-investigation of large-amplitude free vibrations of beams using finite elements, *Journal of Sound and Vibration* 143 (1990) 351–355.
[14] J.A. Schudt, Identification and response of a nonlinear clamped–clamped beam with an axial tensile load: theory and experiment, *Ph.D. Dissertation*, The Ohio State University, 1994.